A computer computes partial sums of the series $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ by adding one term at a time. If each addition requires 10^{-20} seconds, how long do we have to wait to reach 5000?

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Let $S_{N-1} = \sum_{n=3}^{N-1} \frac{\ln n}{n} < 5000$ and $S_N = \sum_{n=3}^{N} \frac{\ln n}{n} \geq 5000$.

$\Rightarrow S_{N-1} < 5000 \leq S_N$

If $f(n) = \frac{\ln n}{n}$, $f'(n) = \frac{\frac{1}{n} n - \ln n \cdot 1}{n^2}$

$= \frac{1-\ln n}{n^2}$

$f(n) > 0$ and $f(n) < 0$ for $n \geq 3$.

$\int_3^N \frac{\ln n}{n} dn$ $\quad u = \ln n$, $du = \frac{1}{n} dn$

$= \int_{\ln 3}^{\ln N} u \, du$

$= \frac{u^2}{2} \Big|_{\ln 3}^{\ln N}$

$= \frac{(\ln N)^2 - (\ln 3)^2}{2}$

$\int_3^N \frac{\ln n}{n} dn < S_{N-1} < \int_3^N \frac{\ln n}{n} dn + f(3)$

$S_{N-1} < S_N < \int_3^N \frac{\ln n}{n} dn + f(3)$

$\Rightarrow \int_3^N \frac{\ln n}{n} dn + f(3) > 5000$

$\frac{(\ln N)^2 - (\ln 3)^2}{2} + \frac{\ln 3}{3} > 5000$

$(\ln N)^2 > 10000 - \frac{2\ln 3}{3} + (\ln 3)^2$

$N > e^{\sqrt{10000 - \frac{2\ln 3}{3} + (\ln 3)^2}}$

$\int_3^N \frac{\ln n}{n} dn < 5000$

$\Rightarrow (\ln N)^2 < 10000 + (\ln 3)^2$

$N < e^{\sqrt{10000 + (\ln 3)^2}}$

$\Rightarrow N \approx e^{\sqrt{10000}} = e^{100} = 2.7 \times 10^{43}$

$\therefore 10^{-20} \times 2.7 \times 10^{43} = \frac{2.7 \times 10^{23}}{3600 \times 24}$

$= 3.125 \times 10^{18}$ days